An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 44 (2002) 317-322
www.elsevier.com/locate/camwa

Threshold of Global Existence for 3-D Attractive Nonlinear Schrödinger Equations under Confined Potentials

YIU-CHUNG HON
Department of Mathematics, City University of Hong Kong
Tat Chee Avenue, Kowloon, Hong Kong, S.A.R.
maychon@cityu.edu.hk

JIAN ZHANG
Department of Mathematics
Sichuan Normal University, Chengdu, 610066, P.R. China
jianzhan@mail.sc.cninfo.net

XU ZHOU
Department of Mathematics, City University of Hong Kong
Tat Chee Avenue, Kowloon, Hong Kong, S.A.R.

(Received February 2001; revised and accepted October 2001)

Abstract—Based on a nonlinear critical index, a threshold of global existence on the L^2-norm of initial data for 3-D attractive nonlinear Schrödinger equations under confined potentials is derived. For numerical verification, a newly developed radial basis function method is applied in this paper to solve the related nonlinear scalar field equation in R^3.

Keywords—Nonlinear Schrödinger equation, Confined potential, Global existence, Radial basis function.

1. INTRODUCTION

Consider the following attractive nonlinear Schrödinger equation under confined potential in three-dimensional space:

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x)\psi + a|\psi|^{p-1}\psi, \qquad t \geq 0, \quad x \in R^3, \tag{1.1}$$

where $i = \sqrt{-1}$ is the imaginary unit, $a < 0$ is an attractive parameter, $p > 1$ is a nonlinear interaction index, $\psi = \psi(t,x)$ is a complex-valued wave function of $(t,x) \in R^+ \times R^3$, and $V(x)$ is a confined potential such that $\inf V > -\infty$.

The work described in this paper was partially supported by a Strategic Research Grant from the City University of Hong Kong (Project No. 7001209).

PII: S0898-1221(02)00150-5

It is well known that when $V(x) = |x|^2$ and $p = 3$, the nonlinear Schrödinger equation (1.1) describes the remarkable Bose-Einstein condensation phenomenon of attractive inter-particle interactions under magnetic trap (see [1] and the references therein). Equation (1.1) has been studied extensively when $V(x)$ is bounded and $1 < p < 5$. For instance, the studies of local well-posedness by Ginibre and Velo [2] and standing waves by Lions [3]. For general unbounded potential $V(x)$ and nonlinear index p, the local well-posedness of equation (1.1) was given by Oh [4] and the standing waves were studied by Rabinowitz [5], Oh [6] and Zhang [7], respectively. A good review can be found from Cazenave's monograph [8] and Sulems' recent monograph [9].

In this paper, the global existence of equation (1.1) is investigated under both bounded and unbounded potential $V(x)$. For general $V(x)$, we first determine the critical value of the nonlinear index p for global existence of equation (1.1) about time t. A threshold of the mass N, which is the L^2-norm of wave function ψ, is then derived for global existence of equation (1.1) about time t. Finally, for the purpose of numerical verification, we apply a newly developed meshless computational radial basis function method (see [10,11]) to compute the threshold of the mass N in R^3. From this we also obtain the numerical value of the best constant for the Gagliardo-Nirenberg inequality in three-dimensional space. It is remarked here that the methodology used in this paper is inspired by the work of Weinstein [12] who introduced the idea of using variational argument to investigate the classical nonlinear Schrödinger equation without any potential.

2. THE CRITICAL VALUE OF p

Under the following initial condition:

$$\psi(0,x) = \psi_0(x), \qquad x \in R^3, \tag{2.1}$$

equations (1.1) and (2.1) become a classical Cauchy problem. An energy space in the course of nature is then defined by

$$H := \left\{ \varphi \in H^1\left(R^3\right), \int V(x)\left|\varphi\right|^2 dx < \infty \right\}. \tag{2.2}$$

For simplicity, we denote $\int_{R^3} \cdot dx$ by $\int \cdot \, dx$. Here, H is a Hilbert space which is continuously embedded in $H^1(R^3)$ with the inner product

$$\langle \varphi, \phi \rangle_H = \int \nabla\varphi\nabla\bar{\phi} + (V - \inf V)\,\varphi\bar{\phi} + \varphi\bar{\phi}\,dx, \tag{2.3}$$

whose associated norm is denoted by $\|\cdot\|_H$.

Let $1 < p < 5$. Define the mass functional

$$N(\varphi) = \int |\varphi|^2 \, dx \tag{2.4}$$

and the energy functional

$$E(\varphi) := \int |\nabla\varphi|^2 + V(x)\left|\varphi\right|^2 + \frac{2a}{p+1}\left|\varphi\right|^{p+1} dx, \qquad \varphi \in H. \tag{2.5}$$

It is clear from the Sobolev's embedding theorem that N and E are well defined. From the point of view of Hamiltonian systems, E is the generating Hamiltonian of equation (1.1).

From Oh [4], we have the following result on the local well posedness of equation (1.1).

LEMMA 2.1. *Let V satisfy that $\inf V > -\infty$ and for each $|\alpha| \geq 2$, $|D^\alpha V|$ is bounded, $1 \leq p < 5$, and $\psi_0 \in H$. For some $T \in [0,\infty)$, the Cauchy problem (1.1) and (2.1) has a unique bounded solution ψ in $C([0,T),H)$. Also, $\psi(t,\cdot)$ about time $t \in [0,T)$ satisfies both conservation laws of mass N: $N(\psi) = N(\psi_0)$ and energy E: $E(\psi) = E(\psi_0)$. Moreover, if T is the maximal existence time of the local solution ψ, then T has the alternatives: either $T = \infty$ (global existence) or $T < \infty$ and $\lim_{t\to T} \|\psi\|_H = \infty$ (collapse).*

Also from Oh [4], we have the following global well-posedness of equation (1.1) when $1 < p < 1 + 4/3$.

PROPOSITION 2.1. *Let V satisfy that $\inf V > -\infty$ and for each $|\alpha| \geq 2$, $|D^\alpha V|$ is bounded, $1 \leq p < 1 + 4/3$, and $\psi_0 \in H$. The Cauchy problem (1.1) and (2.1) has a unique bounded solution ψ in $C([0,\infty), H)$. Also, $\psi(t,\cdot)$ about $t \in [0,\infty)$ satisfies both conservation laws of mass N and energy E.*

When $1 + 4/3 \leq p < 5$, for many potentials $V(x)$, there exist initial data $\psi_0 \in H$ such that the solutions of the Cauchy problem (1.1) and (2.1) collapse in a finite time (see, for examples, [8,9,13]). This is the reason why $p = 1 + 4/3$ is called a critical value of nonlinear index p for global existence of equation (1.1). In the following, under $p = 1 + 4/3$ and for general $V(x)$, we derive a threshold of the mass N for global existence of equation (1.1).

3. THE THRESHOLD OF THE MASS N

To obtain a threshold of the mass N, we introduce the following nonlinear scalar field equation in R^3:

$$-\Delta u + \frac{2}{3}u - |u|^{4/3}u = 0, \qquad u \in H^1(R^3). \tag{3.1}$$

From Weinstein [12] and Kwong [14], we have the following variational results.

LEMMA 3.1. *Equation (3.1) has a unique positive radially symmetric solution $Q(x)$, that is, $Q(x) = Q(|x|)$. Moreover, $(3/5)(\int Q^2\,dx)^{2/3}$ is the minimum of the functional*

$$I(\varphi) = \frac{\left(\int |\nabla\varphi|^2\,dx\right)\left(\int |\varphi|^2\,dx\right)^{2/3}}{\int |\varphi|^{2+4/3}\,dx}, \qquad \varphi \in H^1\left(R^3\right). \tag{3.2}$$

From this lemma, we immediately obtain the following Gagliardo-Nirenberg inequality with the best constant in R^3.

COROLLARY 3.1. *Let Q be the unique positive radially symmetric solution of equation (3.1). Then $\forall\, \varphi \in H^1(R^3)$, we have*

$$\int |\varphi|^{2+4/3}\,dx \leq \frac{5}{3}\left(\int Q^2\,dx\right)^{-2/3}\left(\int |\nabla\varphi|^2\,dx\right)\left(\int |\varphi|^2\,dx\right)^{2/3}. \tag{3.3}$$

In addition,

$$\frac{5}{3}\int |\nabla Q|^2\,dx = \int |Q|^{2+4/3}\,dx. \tag{3.4}$$

The result of the following main theorem is based on the variational arguments given by Weinstein [12].

THEOREM 3.1. *Let V satisfy that $\inf V > -\infty$ and for each $|\alpha| \geq 2$, $|D^\alpha V|$ is bounded and $p = 1 + 4/3$. If ψ_0 satisfies $\psi_0 \in H$ and*

$$\int |\psi_0|^2\,dx < |a|^{-3/2}\int |Q|^2\,dx, \tag{3.5}$$

where Q is the positive radially symmetric solution of equation (3.1), then the solution ψ of the Cauchy problem (1.1) and (2.1) exists globally in time.

PROOF. For $\psi_0 \in H$, let ψ be a solution of the Cauchy problem (1.1) and (2.1) in $C([0,T); H)$ for some $T > 0$. By Lemma 2.1, for $t \in [0,T)$, we have $N(\psi) = N(\psi_0)$ and $E(\psi) = E(\psi_0)$. Since $p = 1 + 4/3$, from (2.5), we have

$$\int |\nabla\varphi|^2 + V(x)\,|\varphi|^2 + \frac{a}{1+2/3}\,|\varphi|^{2+4/3}\,dx = E(\psi_0). \tag{3.6}$$

Since $a < 0$, it follows from (3.3) that

$$\int \left[1 + a \left(\frac{\int |\psi|^2 \, dx}{\int Q^2 \, dx} \right)^{2/3} \right] |\nabla \psi|^2 + V(x) |\psi|^2 \, dx \le E(\psi_0). \tag{3.7}$$

From (2.4), it follows that

$$\int \left[1 + a \left(\frac{\int |\psi|^2 \, dx}{\int Q^2 \, dx} \right)^{2/3} \right] |\nabla \psi|^2 + (V - \inf V) |\psi|^2 \, dx \le E(\psi_0) - N(\psi_0) \inf V. \tag{3.8}$$

Thus, from (3.5) and (3.8), $\|\psi\|_H$ is bounded for $t \in [0, T)$ and some $T > 0$. From Lemma 2.1, the maximal existence time T is equal to ∞. We have then proven that ψ exists globally in time.

REMARK 3.1. *Although the global existence on the solution of the Cauchy problem (1.1) and (2.1) can also be derived from Oh [4] when $\|\psi_0\|_H$ is sufficiently small, Theorem 3.1 gives a different threshold of $\|\psi_0\|_{L^2(R^3)}$ which guarantees the global existence of the solutions.*

For numerical verification, the numerical value of this threshold is computed in the following section by a newly developed meshless computational method.

4. NUMERICAL COMPUTATIONS

In this section, we apply a newly developed radial basis function (RBF) method to compute the value of $\int |Q|^2 \, dx$. By Lemma 3.1, $Q(x) = Q(|x|)$. Putting $|x| = r$, the nonlinear scalar field equation (3.1) becomes

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - \frac{2}{3}u + u|u|^{4/3} = 0, \qquad 0 < r < \infty, \tag{4.1}$$

with boundary conditions

$$\begin{aligned} \frac{du(0)}{dr} &= 0, \\ u(\infty) &= 0, \end{aligned} \tag{4.2}$$

which has a unique positive solution. From the result of the last Section 3, the threshold of the mass N can then be obtained by computing the value

$$\int | Q |^2 \, dx = 4\pi \int_0^\infty r^2 [u(r)]^2 \, dr. \tag{4.3}$$

For the purpose of numerical computation, the infinite solution domain is transformed to [0,1] by using a simple transformation $x = r/(1 + r)$. Equation (4.1) is then transformed to

$$(1-x)^4 \frac{d^2u}{dx^2} + 2(1-x)^3 \left(\frac{1}{x} - 1 \right) \frac{du}{dx} - \frac{2}{3}u + u|u|^{4/3} = 0, \qquad 0 < x < 1, \tag{4.4}$$

with boundary conditions

$$\begin{aligned} \frac{du(0)}{dx} &= 0, \\ u(1) &= 0. \end{aligned} \tag{4.5}$$

The nonlinear equation (4.4) is solved iteratively by

$$(1-x)^4 \frac{d^2}{dx^2} u^n + 2(1-x)^3 \left(\frac{1}{x} - 1 \right) \frac{d}{dx} u^n - \frac{2}{3} u^n = -u^{n-1} \left| u^{n-1} \right|^{4/3}, \qquad 0 < x < 1, \tag{4.6}$$

where $u^n(x)$ denotes the iterated solution $u(x)$ at the iterative step n, $n = 1, 2, \dots$. The idea of the RBF method is to approximate the unknown function $u^n(x)$ by a linear combination of radial basis functions $|x - x_j|^3$ and some polynomial terms as

$$u^n(x) \simeq \sum_{j=1}^{M} \beta_j^n |x - x_j|^3 + \beta_{M+1}^n x + \beta_{M+2}^n, \tag{4.7}$$

where x_j, $j = 1, 2, \dots, M$, are M arbitrary distinct points in (0, 1) and β_j^n are $M+2$ undetermined coefficients to be computed at each iteration n. A direct collocation on another arbitrary M distinct points x_i in (0, 1) by assuming that u^n satisfies equation (4.6) gives M equations as, for $i = 1, 2, \dots, M$,

$$(1 - x_i)^4 \frac{d^2}{dx^2} u^n(x_i) + 2(1 - x_i)^3 \left(\frac{1}{x_i} - 1\right) \frac{d}{dx} u^n(x_i) - \frac{2}{3} u^n(x_i) = -u^{n-1}(x_i) \left|u^{n-1}(x_i)\right|^{4/3}. \tag{4.8}$$

From the boundary equations (4.5), we have the remaining two equations

$$\frac{du^n(0)}{dx} = 0 \tag{4.9}$$

and

$$u^n(1) = 0, \tag{4.10}$$

to obtain a system of $M + 2$ linear equations for the $M + 2$ unknown coefficients βs. Refer to [10,11] for more details on the theoretical foundation and applications of this RBF method. In this paper, the numerical computation is taken by choosing $x_j = j\epsilon$, $x_i = i\epsilon$ where $\epsilon = 1/(M+1)$ and $M = 99$. The initial guess u^0 can be taken to be any positive constant. The iteration will stop until the maximum relative difference between two successive iterated approximations u^n and u^{n-1} at each point x_i is less than a prescribed small value, say 10^{-6} in this paper. The total number of iterations is then 24. It is noted here that the resultant matrix for the linear system is fixed in each iteration. The inverse of the resultant matrix needs only be calculated once by using Gaussian elimination method at the first iteration. The curve of u^{24} is plotted in Figure 1 for illustration. To further compute the threshold of the mass N from equation (4.3), we apply the transformation $x = r/(1 + r)$ and a standard trapezoidal rule to evaluate the following integral:

$$\int |Q|^2 \, dx = 4\pi \int_0^\infty r^2 \left[u^{24}(r)\right]^2 dr = 4\pi \int_0^1 \frac{x^2}{(1 - x)^4} \left[u^{24}(x)\right]^2 dx \approx 63.815. \tag{4.11}$$

Finally, from Theorem 3.1, we obtain the following theorem.

THEOREM 4.1. *Let V satisfy that $\inf V > -\infty$ and for each $|\alpha| \geq 2$, $|D^\alpha V|$ is bounded, $p = 1 + 4/3$. If ψ_0 satisfies $\psi_0 \in H$ and*

$$\int |\psi_0|^2 \, dx < (63.815)|a|^{-3/2},$$

then the solution ψ of the Cauchy problem (1.1) and (2.1) exists globally in time.

From (4.11), we get

$$\frac{5}{3} \left(\int Q^2 \, dx\right)^{-2/3} \approx 0.1044.$$

By using Corollary 3.1, we get the Gagliardo-Nirenberg inequality with the best constant in R^3 as follows.

THEOREM 4.2. *For $\forall \varphi \in H^1(R^3)$, we have*

$$\int |\varphi|^{2+4/3} \, dx \leq (0.1044) \left(\int |\nabla \varphi|^2 \, dx\right) \left(\int |\varphi|^2 \, dx\right)^{2/3}. \tag{4.12}$$

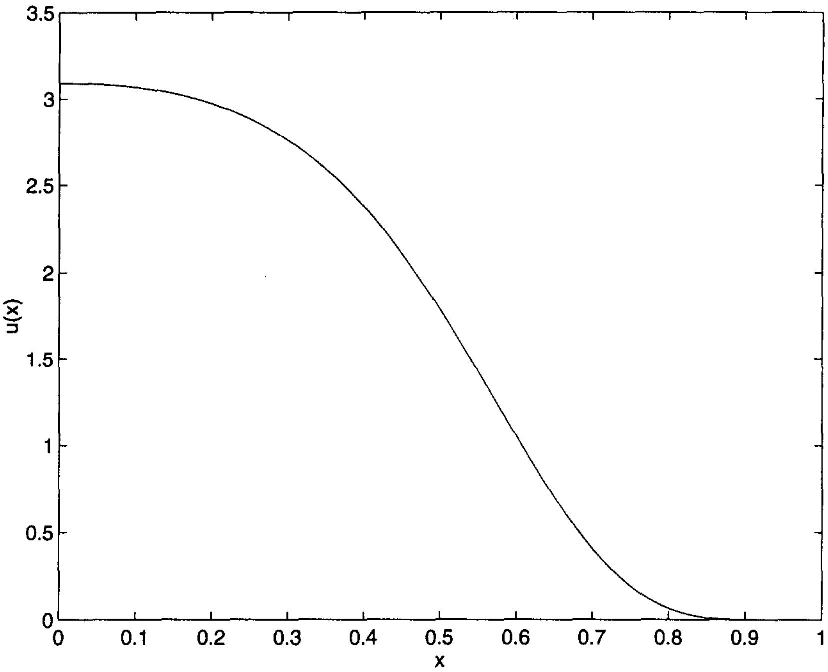

Figure 1. Curve of the approximation $u^{24}(x)$.

REFERENCES

1. F. Dalfovo, S. Giorgini, L.P. Pitaevskii and S. Stringari, Theory of Bose-Einstein condensation in trapped gases, *Reviews of Modern Physics* **71** (3), 463–512, (1999).
2. J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations, *J. Funct. Anal.* **32**, 1–71, (1979).
3. P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. Part I, *Ann. Inst. H. Poincaré. Anal. Non Lineéaire* **1**, 109–145, (1984).
4. Y.G. Oh, Cauchy Problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials, *Journal of Differential Equations* **81**, 255–274, (1989).
5. P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* **43**, 270–291, (1992).
6. Y.G. Oh, Stability of semiclassical bound states of nonlinear Schrödinger equations with potentials, *Commun. Math. Phys.* **121**, 11–33, (1989).
7. J. Zhang, Stability of standing waves for nonlinear Schrödinger equations with unbounded potentials, *Z. Angew. Math. Phys.* **51**, 498–503, (2000).
8. T. Cazenave, An introduction to nonlinear Schrödinger equations, In *Textos de Metodos Matematicos, Volume 22*, Rio de Janeiro, (1989).
9. C. Sulem and P.L. Sulem, *The Nonlinear Schrödinger Equation, Self-Focusing and Wave Collapse*, Springer, (1999).
10. Y.C. Hon, K.F. Cheung, X.Z. Mao and E.J. Kansa, Multiquadric solution for shallow water equations, *ASCE J. Hydraulic Engineering* **125**, 524–533, (1999).
11. C. Franke and R. Schaback, Solving partial differential equations by collocation using radial basis functions, *Appl. Math. Comput.* **93**, 73–82, (1998).
12. M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolations estimates, *Commun. Math. Phys.* **87**, 567–576, (1983).
13. T. Tsurumi and M. Wadati, Collapses of wave functions in multi-dimensional nonlinear Schrödinger equations under harmonic potential, *J. Phys. Soc. Jpn.* **66**, 3031–3034, (1997).
14. M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^N, *Arch. Rat. Mech. Anal.* **105**, 243–266, (1989).